ANALYTICAL GEOMETRY IN THREE DIMENSIONS

PRECALCULUS

CHAPTER 11

- This Slideshow was developed to accompany the textbook
  - *Precalculus*
  - *By Richard Wright*
  - *https://www.andrews.edu/~rwright/Precalculus-RLW/Text/TOC.html*
- Some examples and diagrams are taken from the textbook.

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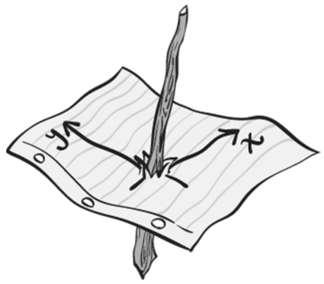
11-01 3-D COORDINATE SYSTEM

IN THIS SECTION, YOU WILL:

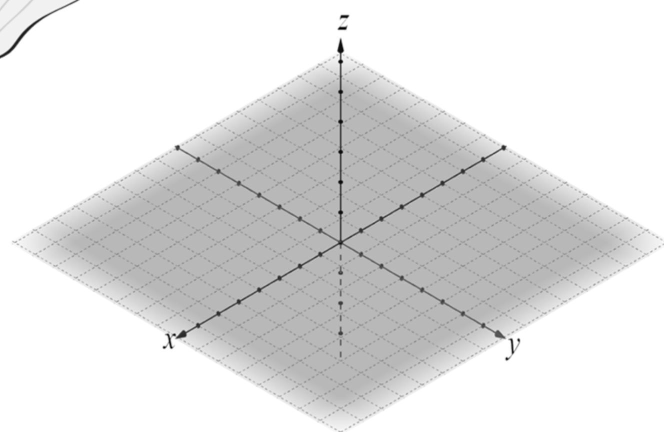
- PLOT A POINT IN 3-DIMENSIONS.
- CALCULATE 3-DIMENSIONAL DISTANCE AND MIDPOINT.
- FIND AND GRAPH THE EQUATION OF A SPHERE.
- FIND A TRACE OF A SPHERE.

11-01 3-D COORDINATE SYSTEM

- Points in 3 dimensions
  - (x, y, z)



- Graph by moving out the *x*, over the *y*, then up the *z*.
  - Graph A(5, 6, 3)
  - Graph B(-2, -4, 0)



11-01 3-D COORDINATE SYSTEM

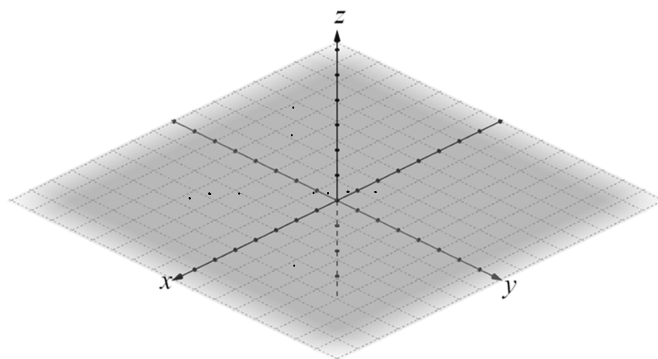
- Distance Formula
  - In 2-D:
    - $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
  - In 3-D: (just add the z)
    - $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

11-01 3-D COORDINATE SYSTEM

- Midpoint Formula
  - In 2-D:
    - $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
  - In 3-D: (just add the z)
    - $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

11-01 3-D COORDINATE SYSTEM

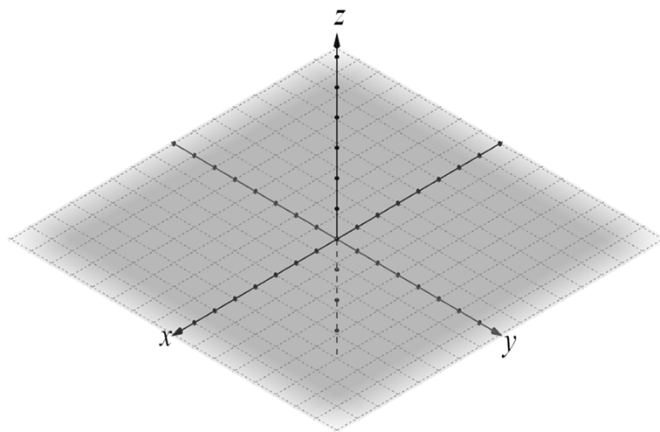
- Equation of Circle (2-D)
  - $(x-h)^2 + (y-k)^2 = r^2$
- Equation of Sphere (3-D) (just add z)
  - $(x-h)^2 + (y-k)^2 + (z-j)^2 = r^2$
  - Center is (h, k, j), r = radius
  - Graph by plotting the center and moving each direction the radius
  - Graph $(x-2)^2 + (y+1)^2 + (z+1)^2 = 16$



Center (2, -1, -1)
$r^2 = 16$ so $r = 4$

11-01 3-D COORDINATE SYSTEM

- Trace (like intercepts for a sphere)
  - Draw the *xy* trace for
    $(x-2)^2+(y+1)^2+(z+1)^2=16$



Since *xy* trace, let $z=0$

$$(x-2)^2+(y+1)^2+(1)^2=16$$
$$(x-2)^2+(y+1)^2=15$$

Center (2, -1)

$$r=\sqrt{15}\approx 3.9$$

Looks funny because a perspective drawing

11-02 VECTORS IN SPACE

IN THIS SECTION, YOU WILL:

- USE VECTOR OPERATIONS IN THREE DIMENSIONS.
- FIND THE ANGLE BETWEEN VECTORS.

11-02 VECTORS IN SPACE

- Vectors in 2-D
  - $\vec{v} = \langle v_1, v_2 \rangle$
- Vectors in 3-D (just add z)
  - $\vec{v} = \langle v_1, v_2, v_3 \rangle$
- To find a vector from the initial point (p_1, p_2, p_3) to the terminal point (q_1, q_2, q_3)
  - $\vec{v} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$
- If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{u} = \langle u_1, u_2, u_3 \rangle$,
- Addition
  - Add corresponding elements
  - $\vec{v} + \vec{u} = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$
- Scalar multiplication
  - Distribute
  - $c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle$

11-02 VECTORS IN SPACE

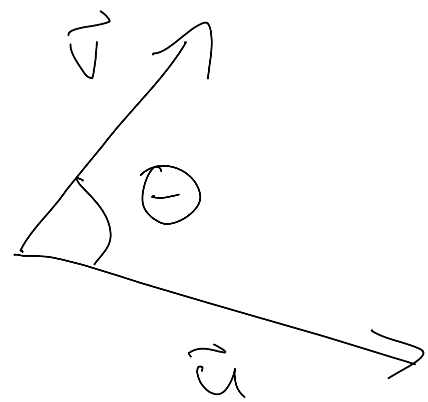
- If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{u} = \langle u_1, u_2, u_3 \rangle$,

- Dot Product
  - $\vec{v} \cdot \vec{u} = v_1 u_1 + v_2 u_2 + v_3 u_3$
- Magnitude
  - $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
- Unit vector in the direction of $\vec{v}$
  - $\frac{\vec{v}}{\|\vec{v}\|}$

11-02 VECTORS IN SPACE

- Angle between vectors
  - $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$

  - If $\theta = 90°$ (and $\vec{u} \cdot \vec{v} = 0$)
    - Then vectors are orthogonal
  - If $\vec{u} = c\vec{v}$
    - Then vectors are parallel



11-02 VECTORS IN SPACE

- Let $\vec{m} = \langle 1, 0, 3 \rangle$ and $\vec{n} = \langle -2, 1, -4 \rangle$
- Find $\|\vec{m}\|$
- Find unit vector in direction of $\vec{m}$
- Find $\vec{m} + 2\vec{n}$

$$\|\vec{m}\| = \sqrt{m_1^2 + m_2^2 + m_3^2}$$
$$= \sqrt{1^2 + 0^2 + 3^2}$$
$$= \sqrt{10}$$

$$\frac{\vec{m}}{\|\vec{m}\|} = \frac{\langle 1, 0, 3 \rangle}{\sqrt{10}}$$
$$= \left\langle \frac{1}{\sqrt{10}}, \frac{0}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$
$$= \left\langle \frac{\sqrt{10}}{10}, 0, \frac{3\sqrt{10}}{10} \right\rangle$$

$$\langle 1, 0, 3 \rangle + 2\langle -2, 1, -4 \rangle$$
$$\langle 1, 0, 3 \rangle + \langle -4, 2, -8 \rangle$$
$$\langle -3, 2, -5 \rangle$$

11-02 VECTORS IN SPACE

- Let $\vec{m} = \langle 1, 0, 3\rangle$ and $\vec{n} = \langle -2, 1, -4\rangle$
- Find $\vec{m} \cdot \vec{n}$
- Find the angle between $\vec{m}$ and $\vec{n}$

$$\langle 1, 0, 3\rangle \cdot \langle -2, 1, -4\rangle$$
$$1(-2) + 0(1) + 3(-4)$$
$$-14$$

$$\vec{m} \cdot \vec{n} = \|\vec{m}\|\|\vec{n}\| \cos\theta$$
$$-14 = \sqrt{1^2 + 0^2 + 3^2}\sqrt{(-2)^2 + 1^2 + (-4)^2} \cos\theta$$
$$-14 = \sqrt{10}\sqrt{21} \cos\theta$$
$$\frac{-14}{\sqrt{10}\sqrt{21}} = \cos\theta$$
$$\theta \approx 165.0°$$

11-02 VECTORS IN SPACE

- Are $\vec{p} = \langle 1, 5, -2 \rangle$ and $\vec{q} = \left\langle -\frac{1}{5}, -1, \frac{2}{5} \right\rangle$ parallel, orthogonal, or neither?

- Parallel if $\vec{p} = c\vec{q}$

Orthogonal if $\vec{p} \cdot \vec{q} = 0$

$$\langle 1, 5, -2 \rangle \cdot \left\langle -\frac{1}{5}, -1, \frac{2}{5} \right\rangle$$

$$1\left(-\frac{1}{5}\right) + 5(-1) + (-2)\left(\frac{2}{5}\right)$$

$$-\frac{1}{5} - 5 - \frac{4}{5} = -6$$

Not 0, so not orthogonal

$$\langle 1, 5, -2 \rangle = c\left\langle -\frac{1}{5}, -1, \frac{2}{5} \right\rangle$$

Check *x*

$$1 = c\left(-\frac{1}{5}\right) \rightarrow c = -5$$

Check *y*

$$5 = c(-1) \rightarrow c = -5$$

Check *z*

$$-2 = c\left(\frac{2}{5}\right) \to c = -5$$

c is always the same, so they are parallel

11-02 VECTORS IN SPACE

- Are $P(1, -1, 3)$, $Q(0, 4, -2)$, and $R(6, 13, -5)$ collinear?

Find $\overrightarrow{PQ}$ and $\overrightarrow{QR}$. If they are parallel, then they go in same direction.
Since they would share a point, then they would be the same line.

$$\begin{aligned}\overrightarrow{PQ} &= \langle 0-1, 4-(-1), -2-3\rangle \\ &= \langle -1, 5, -5\rangle \\ \overrightarrow{QR} &= \langle 6-0, 13-4, -5-(-2)\rangle \\ &= \langle 6, 9, -3\rangle\end{aligned}$$

These are not parallel because $\overrightarrow{PQ} \neq c\overrightarrow{QR}$
They are not going same direction, so not collinear

11-03 CROSS PRODUCTS

IN THIS SECTION, YOU WILL:

- EVALUATE A CROSS PRODUCT.
- USE A CROSS PRODUCT TO SOLVE AREA AND VOLUME PROBLEMS.

11-03 CROSS PRODUCTS

- $\hat{\imath}$ is unit vector in x, $\hat{\jmath}$ is unit vector in y, and $\hat{k}$ is unit vector in z
- $\vec{u} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}$ and $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$
- $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- If $\vec{u} = \langle -2, 3, -3 \rangle$ and $\vec{v} = \langle 1, -2, 1 \rangle$, find $\vec{u} \times \vec{v}$

$$\vec{u} \times \vec{v} = \left| \begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 3 & -3 \\ 1 & -2 & 1 \end{matrix} \right| \begin{matrix} \hat{\imath} & \hat{\jmath} \\ -2 & 3 \\ 1 & -2 \end{matrix}$$

$$= 3\hat{\imath} + (-3)\hat{\jmath} + 4\hat{k} - 3\hat{k} - 6\hat{\imath} - (-2)\hat{\jmath}$$

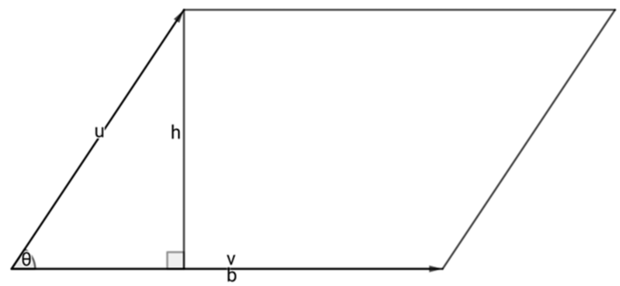
$$= -3\hat{\imath} - \hat{\jmath} + \hat{k} = \langle -3, -1, 1 \rangle$$

11-03 CROSS PRODUCTS

- Properties of Cross Products
- $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
- $c(\vec{u} \times \vec{v}) = c\vec{u} \times \vec{v} = \vec{u} \times c\vec{v}$
- $\vec{u} \times \vec{u} = 0$
  - If $\vec{u} \times \vec{v} = 0$, then $\vec{u}$ and $\vec{v}$ are parallel
- $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
- $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$
- $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$

11-03 CROSS PRODUCTS

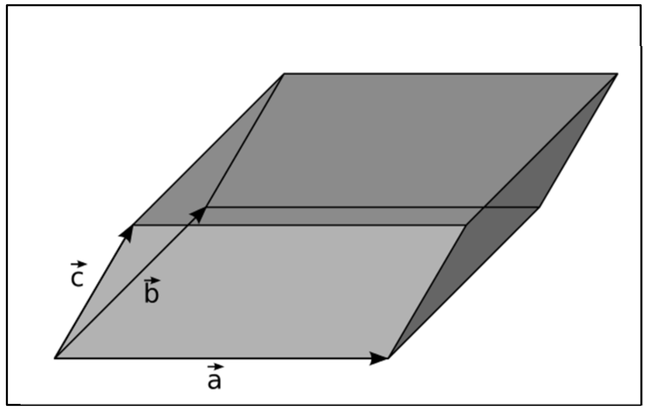
- $A = bh$
- $h = \|\vec{u}\| \sin\theta$
- $A = \|\vec{v}\|\|\vec{u}\| \sin\theta$
- Area of a Parallelogram
  - $\|\vec{u} \times \vec{v}\|$ where $\vec{u}$ and $\vec{v}$ represent adjacent sides



11-03 CROSS PRODUCTS

- Triple Scalar Product (shortcut)
- $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

- Volume of Parallelepiped
  - (3-D parallelogram)
  - $V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$ where $\vec{u}$, $\vec{v}$, and $\vec{w}$ represent adjacent edges



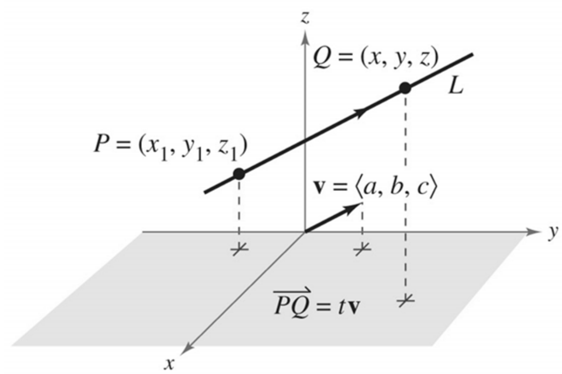
11-04 LINES AND PLANES IN SPACE

IN THIS SECTION, YOU WILL:

- WRITE AN EQUATION FOR A LINE IN THREE DIMENSIONS.
- WRITE AN EQUATION FOR A PLANE.
- FIND THE ANGLE BETWEEN TWO PLANES.
- GRAPH A PLANE.

11-04 LINES AND PLANES IN SPACE

- Lines
  - Line L goes through points P and Q
  - $\vec{v}$ is a direction vector for L
  - Start at P and move any distance in direction $\vec{v}$ to get some point Q
  - $\overrightarrow{PQ} = t\vec{v}$ because they are parallel
  - $\langle x - x_1, y - y_1, z - z_1 \rangle = \langle at, bt, ct \rangle$
    - General form



11-04 LINES AND PLANES IN SPACE

- Parametric Equations of Line
  - Take each component of the general form and solve for x, y, or z.
    $x = at + x_1$
  - $y = bt + y_1$
    $z = ct + z_1$
  - We used these when we solved 3-D systems of equations and got many solutions
- Symmetric Equation of Line
  - Solve each equation in parametric equations for t
  - $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

11-04 LINES AND PLANES IN SPACE

- Find a set of parametric equations of the line that passes through (1, 3, -2) and (4, 0, 1).

Find the direction vector between those two points.

$$\vec{v} = \langle 4-1, 0-3, 1-(-2) \rangle$$
$$= \langle 3, -3, 3 \rangle$$
$$= \langle a, b, c \rangle$$

Let's call the first point $(1, 3, -2) = (x_1, y_1, z_1)$

Plug it in

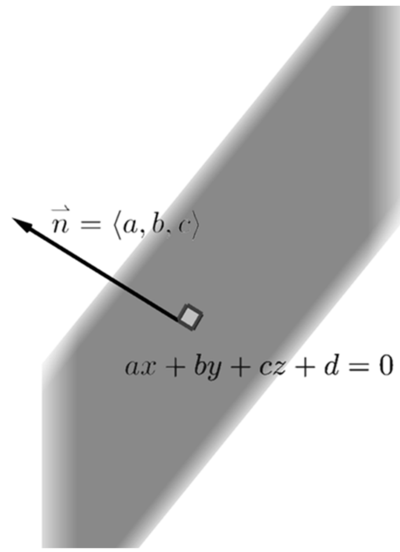
$$x = at + x_1$$
$$y = bt + y_1$$
$$z = ct + z_1$$
$$x = 3t + 1$$
$$y = -3t + 3$$
$$z = 3t - 2$$

11-04 LINES AND PLANES IN SPACE

- Planes
- $\overrightarrow{PQ} \cdot \vec{n} = 0$ because they are perpendicular

- Standard form
- $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

- General form
- $ax + by + cz + d = 0$



11-04 LINES AND PLANES IN SPACE

- Find the general equation of plane passing through $A(3,2,2)$, $B(1,5,0)$, and $C(1,-3,1)$

We need to find the normal vector to the plane.

Find two vectors in the plane

$$\overrightarrow{AB} = \langle 1-3, 5-2, 0-2 \rangle = \langle -2, 3, -2 \rangle$$

$$\overrightarrow{BC} = \langle 1-1, -3-5, 1-0 \rangle = \langle 0, -8, 1 \rangle$$

Find the cross product to get a perpendicular (normal) vector

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{BC}$$

$$\vec{n} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 3 & -2 \\ 0 & -8 & 1 \end{vmatrix} \begin{matrix} \vec{\imath} & \vec{\jmath} \\ -2 & 3 \\ 0 & -8 \end{matrix}$$

$$= 3\vec{\imath} + 0\vec{\jmath} + 16\vec{k} - 0\vec{k} - 16\vec{\jmath} - 2\vec{\jmath}$$

$$= -13\vec{\imath} + 2\vec{\jmath} + 16\vec{k} = \langle a, b, c \rangle$$

Fill in the general form

I chose $B(1,5,0) = (x_1, y_1, z_1)$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

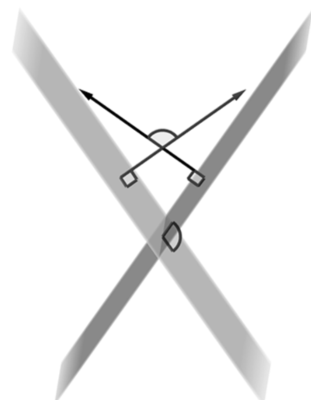
$$-13(x-1) + 2(y-5) + 16(z-0) = 0$$

Simplify to get general form

$$-13x + 2y + 16z + 3 = 0$$

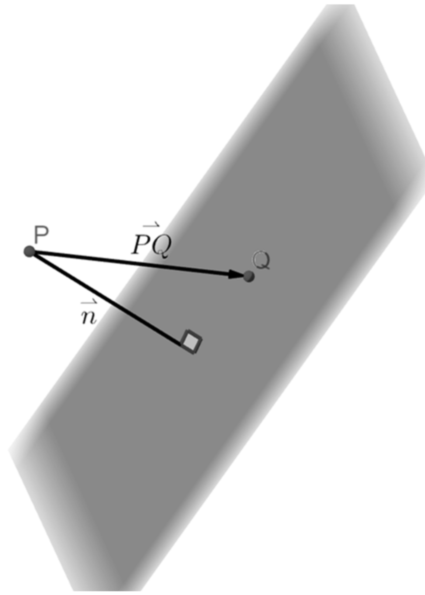
11-04 LINES AND PLANES IN SPACE

- Angle between two planes
  - Find the angle between normal vectors
  - Normal vectors are coefficients in the equations of the plane
  - $|\overrightarrow{n_1} \cdot \overrightarrow{n_2}| = \|\overrightarrow{n_1}\| \|\overrightarrow{n_2}\| \cos\theta$



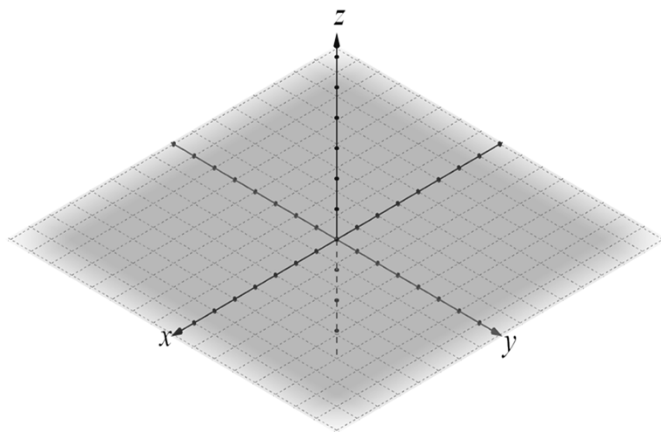
11-04 LINES AND PLANES IN SPACE

- Distance between a Point and a Plane
- $D = \left\| proj_{\vec{n}}\, \overrightarrow{PQ} \right\|$
- $D = \frac{\left|\overrightarrow{PQ} \cdot \vec{n}\right|}{\|\vec{n}\|}$



11-04 LINES AND PLANES IN SPACE

- Graphing planes in space
  - Find the intercepts
  - Plot the intercepts
  - Draw a triangle to represent the plane
- Sketch $3x + 4y + 6z = 24$



x-int $3x = 24 \rightarrow x = 8$
y-int $4y = 24 \rightarrow y = 6$
z-int $6z = 24 \rightarrow z = 4$